

Second-Order Interpolation Conditions for Univariate Functions with Lipschitz Continuous Hessian, and Application to Automated Performance Estimation

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1 Introduction

The *Performance Estimation Problem* (PEP) framework was introduced to automatically compute worst-case guarantees on the performance of a given optimization method when applied to a given function class \mathcal{F} , by formulating the search for the worst-case function of \mathcal{F} as an optimization problem [1]. To render this infinite-dimensional problem tractable and convex, we only consider the finite set S of iterates and minimizer, and impose conditions on S to ensure its consistency with \mathcal{F} .

Hence, the PEP framework relies on *interpolation conditions*, i.e. necessary and sufficient conditions that a data set S must satisfy to ensure the existence of a function of \mathcal{F} , defined on the whole space and that interpolates S .

2 Second-order interpolation conditions

Presently, the only existing interpolation conditions are first-order conditions, i.e. conditions involving only function and subgradient values, hence confining the PEP framework to the analysis of first-order methods. We propose second-order interpolation conditions for the class \mathcal{H}_M of smooth univariate functions with M -Lipschitz continuous Hessian, i.e. functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f''(x) - f''(y)| \leq M|x - y| \quad \forall x, y \in \mathbb{R}. \quad (1)$$

We define \mathcal{H}_M -interpolability as:

Definition 1 The set $\{(x_i, f_i, g_i, h_i)\}_{i=1,\dots,N} \in \mathbb{R}^{4 \times N}$ is \mathcal{H}_M -interpolable, if and only if,

$$\exists f \in \mathcal{H}_M : \begin{cases} f(x_i) &= f_i \\ f'(x_i) &= g_i \\ f''(x_i) &= h_i \end{cases} \quad \forall i = 1, \dots, N \quad (2)$$

Our main result provides interpolation conditions for \mathcal{H}_M .

Theorem 1 The set $\{(x_i, f_i, g_i, h_i)\}_{i=1,\dots,N}$ is \mathcal{H}_M -interpolable, if and only if, $\forall i, j = 1, \dots, N$,

$$\begin{cases} |\Delta h_{ij}| \leq M|\Delta x_{ij}| \\ T_{ij}^f \geq -\frac{M}{6}|\Delta x_{ij}|^3 + \frac{(T_{ij}^g + \frac{M}{2}\Delta x_{ij}|\Delta x_{ij}|)^2}{2(\Delta h_{ij} + M|\Delta x_{ij}|)} + \frac{(\Delta h_{ij} + M|\Delta x_{ij}|)^3}{96M^2} \end{cases} \quad (3)$$

where $\Delta x_{ij} = x_j - x_i$, $T_{ij}^f = f_j - f_i - g_i \Delta x_{ij} - \frac{h_i}{2} \Delta x_{ij}^2$, $T_{ij}^g = g_j - g_i - h_i \Delta x_{ij}$, and $\Delta h_{ij} = h_j - h_i$.

3 Application to Performance Estimation Problem

With a second-order interpolation condition, we can consider analyzing the worst-case performance of second-order methods, e.g. Newton's method. Since (3) cannot be formulated as convex semidefinite constraints as done classically in the PEP framework [2], our conditions cannot be directly added to this framework. However, it is still possible to use (3) in a non-convex formulation and to obtain tight guarantees via non-convex solvers.

We are thus able to numerically compute the worst-case performance of Newton's method applied to \mathcal{H}_M . More precisely, we show that the guarantee derived in [3, Theorem 1.2.5] is actually tight, and observe that the worst-case function remains the same when changing the performance criterion to objective function values.

4 Perspectives

Future work includes generalization of these second-order conditions to n -variable functions, with $n \geq 2$. Next, a challenge would be to formulate these n -dimensional conditions in a tractable way, that will allow to efficiently solve the associated PEP. This would allow then provide guarantees on second-order methods independently of a function's number of variables, in the spirit of what the PEP framework achieves for first-order methods.

References

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