Second-Order Interpolation Conditions for Univariate Functions with Lipschitz Continuous Hessian, and Application to Automated Performance Estimation

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1 Introduction

The *Performance Estimation Problem* (PEP) framework was introduced to automatically compute worst-case guarantees on the performance of a given optimization method when applied to a given function class $\mathscr{F}$, by formulating the search for the worst-case function of $\mathscr{F}$ as an optimization problem [1]. To render this infinite-dimensional problem tractable and convex, we only consider the finite set S of iterates and minimizer, and impose conditions on S to ensure its consistency with $\mathscr{F}$.

Hence, the PEP framework relies on *interpolation conditions*, i.e. necessary and sufficient conditions that a data set S must satisfy to ensure the existence of a function of $\mathscr{F}$, defined on the whole space and that interpolates S.

2 Second-order interpolation conditions

Presently, the only existing interpolation conditions are first-order conditions, i.e. conditions involving only function and subgradient values, hence confining the PEP framework to the analysis of first-order methods. We propose second-order interpolation conditions for the class $\mathscr{H}_M$ of smooth univariate functions with M-Lipschitz continuous Hessian, i.e. functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$|f''(x) - f''(y)| \leq M|x-y| \quad \forall x, y \in \mathbb{R}. \tag{1}$$

We define $\mathscr{H}_M$-interpolability as:

Definition 1 *The set* $\{(x_i, f_i, g_i, h_i)\}_{i=1,\dots,N} \in \mathbb{R}^{4\times N}$ *is* $\mathscr{H}_M$-*interpolable, if and only if,*

$$\exists f \in \mathscr{H}_M : \begin{cases} f(x_i) &= f_i \\ f'(x_i) &= g_i \\ f''(x_i) &= h_i \end{cases} \quad \forall i = 1,\dots,N \tag{2}$$

Our main result provides interpolation conditions for $\mathscr{H}_M$.

Theorem 1 *The set* $\{(x_i, f_i, g_i, h_i)\}_{i=1,\dots,N}$ *is* $\mathscr{H}_M$-*interpolable, if, and only if,* $\forall i, j = 1, \dots, N$,

$$\begin{cases} |\Delta h_{ij}| \leq M|\Delta x_{ij}| \\ T^f_{ij} \geq -\frac{M}{6}|\Delta x_{ij}|^3 + \frac{\left(T^g_{ij}+\frac{M}{2}\Delta x_{ij}|\Delta x_{ij}|\right)^2}{2\left(\Delta h_{ij}+M|\Delta x_{ij}|\right)} + \frac{\left(\Delta h_{ij}+M|\Delta x_{ij}|\right)^3}{96M^2} \end{cases} \tag{3}$$

where $\Delta x_{ij} = x_j - x_i$, $T^f_{ij} = f_j - f_i - g_i\Delta x_{ij} - \frac{h_i}{2}\Delta x^2_{ij}$, $T^g_{ij} = g_j - g_i - h_i\Delta x_{ij}$, *and* $\Delta h_{ij} = h_j - h_i$.

3 Application to Performance Estimation Problem

With a second-order interpolation condition, we can consider analyzing the worst-case performance of second-order methods, e.g. Newton's method. Since (3) cannot be formulated as convex semidefinite constraints as done classically in the PEP framework [2], our conditions cannot be directly added to this framework. However, it is still possible to use (3) in a non-convex formulation and to obtain tight guarantees via non-convex solvers.

We are thus able to numerically compute the worst-case performance of Newton's method applied to $\mathscr{H}_M$. More precisely, we show that the guarantee derived in [3, Theorem 1.2.5] is actually tight, and observe that the worst-case function remains the same when changing the performance criterion to objective function values.

4 Perspectives

Future work includes generalization of these second-order conditions to n-variable functions, with $n \geq 2$. Next, a challenge would be to formulate these n-dimensional conditions in a tractable way, that will allow to efficiently solve the associated PEP. This would allow then provide guarantees on second-order methods independently of a function's number of variables, in the spirit of what the PEP framework achieves for first-order methods.

References

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