# INTERPOLATION CONDITIONS FOR LINEAR OPERATORS AND APPLICATIONS TO PERFORMANCE ESTIMATION PROBLEMS


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Thesis supported by a FRIA grant


20th EUROpt Workshop, Budapest, 25, August 2023

Worst-case **performance** of an optimization **method** $\mathcal{M}$ on

$$\min_x f(x)$$

where $f \in \mathcal{F}$ has some properties (smoothness, convexity, ...)?

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**Example:** Worst-case performance of $\overbrace{\text{gradient method}}^{\mathcal{M}}$ on $\overbrace{L\text{-smooth convex functions}}^{\mathcal{F}}$ after $N$ iterations?

$$\overbrace{f(x_N) - f^*}^{\text{performance}} \leq \frac{L}{2} \frac{1}{2N+1}$$

Worst-case **performance** of an optimization **method** $\boldsymbol{\mathcal{M}}$ on

$$\min_x f(x) + g(Mx)$$

where $\boldsymbol{f} \in \boldsymbol{\mathcal{F}}$ has some properties (smoothness, convexity, ...)?

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**In this work**

**Worst-case performance of methods involving linear operators?**

(Chambolle-Pock method, Condat-Vũ method, PDFP, PD30, PAPC, ADMM, etc.)

# Performance Estimation Problem (PEP)

Theoretical and practical framework to analyze exact performance of optimization methods on problem classes.

- [Drori and Teboulle, 2014] «Performance of first-order methods for smooth...»
- [Taylor, 2017] «Convex interpolation and performance estimation of first-order...»

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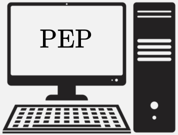
Method
+
Class of functions

→

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Exact worst-case performance

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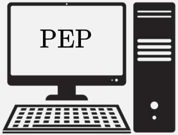
Gradient method

+

Class of $L$-smooth convex functions




Exact worst-case performance

$f(x_N) - f^* \leq \frac{L}{2}\frac{1}{2N+1}$

# PEP IS WELL DEVELOPED: AROUND 40 RESEARCHERS

## Analysis of methods and frameworks

- Decentralized optimization [Colla, Hendrickx, 2022]
- Halper iteration *[Lieder, 2021]*
- Maximal monotone inclusion *[Gu, Yang, 2020]*
- Proximal GM *[Taylor, Hendrickx, Glineur, 2018]*
- Splitting methods *[Ryu, Taylor, Bergeling, Giselsson 2020]*
- Non linear conjugate gradients [Gupta, Freund, Sun, Taylor, 2023]
- Monotone variational inequality *[Gu and Yang, 2022]*
- GM on smooth hypoconvex *[Rotaru, Glineur, Patrinos, 2022]*
- Stochastic optimization *[Taylor, Bach, 2021]*
- Difference of convex method *[Abbaszadehpeivasti, de Klerk, Zamani, 2021]*
- ADMM [Abbaszadehpeivasti, de Klerk, Zamani, 2022]
- GM on smooth strongly convex *[Taylor, Hendrickx, Glineur, 2017]*
- GM on smooth convex *[Drori and Teboulle, 2014]*
- GM exact line search *[de Klerk, Glineur, Taylor, 2017]*
- Extragradient method *[Gorbunov, Loizou, Gidel, 2020]*
- Coordinate descent method *[Shi, Liu, 2016]* *[Kamri, Hendrickx, Glineur, 2022]* *[Abbaszadehpeivasti, de Klerk, Zamani, 2022]*
- Continuous-time models *[Moucer, Taylor, Bach, 2022]*
- Gradient descent-ascent method [Zamani, de Klerk, Abbaszadehpeivasti, 2022]
- PESTO *[Taylor, Hendrickx, Glineur, 2017]*
- Inexact GM *[de Klerk, Glineur, Taylor, 2020]*
- PEPit *[Goujaud, Moucer, Glineur, Hendrickx, Taylor, Dieuleveut, 2022]*
- Methods on convex functions with quadratic upper bound *[Goujaud, Taylor, Dieuleveut, 2022]*
- Bregman methods *[Dragomir, Taylor, d'Aspremont, 2021]*
- GM on functions with lower restricted secant inequality and upper error bound *[Guille-Escuret, Ibrahim, Goujaud, Mitliagkas, 2022]*

## Optimization of methods

- Optimal GM for smooth strongly convex *[Taylor, Drori, 2022]*
- Optimal methods for non-smooth and smooth convex *[Drori, Taylor, 2020]*
- Fast proximal point method *[Kim, 2021]*
- Kelley's cutting-plane method *[Drori, Teboulle, 2016]*
- Optimized GM for smooth convex *[Kim, Fessler, 2016]*
- FISTA *[Kim, Fessler, 2018]*
- Optimal first-order methods for convex and nonconvex *[Gupta, Van Parys, Ryu, 2022]*

GM = Gradient method ADMM = Alternating Directions Method of Multipliers

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- PEPit *[Goujaud, Moucer, Glineur, Hendrickx, Taylor, Dieuleveut, 2022]*
- **Functions involving linear mappings** *[Bousselmi, Hendrickx, Glineur, 2023]*
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# Performance Estimation Problem

# Input = optimization method + problem class
# Output = worst instance in problem class

Example:

- $N$ steps of gradient method $x_{i+1} = x_i - \frac{1}{L}\nabla f(x_i)$
- $L$-smooth convex functions $f$

**PEP**

$$
\begin{aligned}
\max_{\text{points } x_i, x^*, \text{ function } f} \quad & f(x_N) - f(x^*) \\
\text{s.t.} \quad & f \; L\text{-smooth convex}, \\
& x_{i+1} = x_i - \frac{1}{L}\nabla f(x_i), \\
& ||x^* - x_0||^2 \leq 1, \\
& ||\nabla f(x^*)||^2 = 0.
\end{aligned}
$$

## INPUT = OPTIMIZATION METHOD + PROBLEM CLASS
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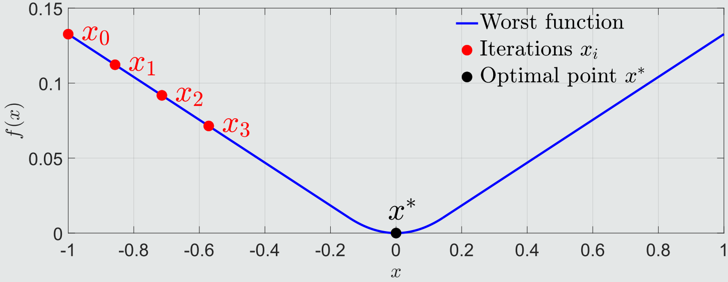
**PEP solved outputs (for $N = 3$ and $L = 1$)**

- Worst performance: $\boldsymbol{f(x_N) - f^* = \frac{1}{14}}(= \frac{L}{2}\frac{1}{2N+1})$
- Worst function:



# PEP as finite-dimensional problem

$f$ **infinite-dimensional** but only access to $x_i$, $f(x_i)$, $\nabla f(x_i)$ ... **black-box** property!

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**PEP**

$$\max_{\text{points } x_i, x^*, f_i, f^*, g_i, g^*} \quad f_N - f^*$$

$$\begin{aligned}
\text{s.t.} \quad & \exists f \ L\text{-smooth convex}: \quad f(x_i) = f_i, \quad \nabla f(x_i) = g_i, \\
& \qquad\qquad\qquad\qquad f(x^*) = f^*, \quad \nabla f(x^*) = g^*, \\
& x_{i+1} = x_i - \frac{1}{L} g_i, \\
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- **Reformulation 1:** Explicit interpolation conditions;

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- **Reformulation 1:** Explicit interpolation conditions;
- **Reformulation 2:** Lift to **convex semidefinite program** (efficiently solvable!)

## Interpolation conditions for $L$-smooth convex functions

Given $\left\{(x_1, g_1, f_1), \ldots, (x_N, g_N, f_N)\right\}$,

$\exists$ $L$-smooth convex $f$ such that $\begin{cases} f(x_i) & = f_i, \ \ \forall i \\ \nabla f(x_i) & = g_i, \ \ \forall i \end{cases}$ if, and only if,

$$f_i \geq f_k + g_k^T(x_i - x_k) + \frac{1}{2L}||g_i - g_k||^2 \quad \forall (i, k).$$

We say that $\left\{(x_1, g_1, f_1), \ldots, (x_N, g_N, f_N)\right\}$ is $L$-smooth-convex-interpolable.

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We say that $\left\{(x_1, g_1, f_1), \dots, (x_N, g_N, f_N)\right\}$ is **$L$-smooth-convex-interpolable**.

**Interpolation conditions** must be convex in $f_i$, $x_i^T g_k$, $x_i^T x_k$, $g_i^T g_k$.
(Otherwise, we cannot solve the PEP for the class of interest).

# Performance Estimation Problem for linear operators

**Goal:** Extend and exploit PEP to analyze methods applied to $\min_x F(x)$ where $F$ involves linear operators.

| $F(x)$ | Possible method | Iterations |
|---|---|---|
| $g(Mx)$ | Gradient descent | $x_{i+1} = x_i - \frac{h}{L} M^T \nabla g(Mx_i)$ |
| $\frac{1}{2} x^T Q x \; (= \frac{1}{2} \|\|Q^{\frac{1}{2}} x\|\|^2)$ | Gradient descent | $x_{i+1} = x_i - \frac{h}{L} Q x_i$ |
| $f(x) + g(Mx)$ | Chambolle-Pock [Chambolle and Pock, 2011] | $\begin{cases} x_{i+1} & = \mathrm{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} & = \mathrm{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$ |

What is missing in PEP to analyze such problems?

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What is missing in PEP to analyze such problems?

**Interpolation conditions** for linear operators!

# Decomposing the iteration

- Gradient method on $\min_x g(Mx)$: $x_{i+1} = x_i - \frac{h}{L} M^T \nabla g(Mx_i)$

- Gradient method on $\min_x g(Mx)$: $x_{i+1} = x_i - \frac{h}{L} M^T \nabla g(Mx_i)$ or equivalently

$$\begin{cases} y_i & = Mx_i \\ u_i & = \nabla g(y_i) \\ v_i & = M^T u_i \\ x_{i+1} & = x_i - \frac{h}{L} v_i \end{cases}$$

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$$\begin{cases} y_i & = Mx_i & \text{New interpolation conditions} \\ u_i & = \nabla g(y_i) & \text{Standard interpolation conditions} \\ v_i & = M^T u_i & \text{New interpolation conditions} \\ x_{i+1} & = x_i - \frac{h}{L} v_i & \text{Standard} \end{cases}$$

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- Chambolle-Pock method on $\min_x f(x) + g(Mx)$ :

$$\begin{cases} x_{i+1} & = \operatorname{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} & = \operatorname{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$$

Also requires New interpolation conditions.

**Definition ($\mathcal{L}_L$-interpolability)**

$\left\{(x_1, y_1), \ldots, (x_N, y_N)\right\}$ and $\left\{(u_1, v_1), \ldots, (u_N, v_N)\right\}$ are **$\mathcal{L}_L$-interpolable** if, and only if,

$$\exists M \text{ with } \sigma_{\max}(M) \leq L \ : \ \begin{cases} y_i = Mx_i, & \forall i, \\ v_i = M^T u_i, & \forall i. \end{cases}$$

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**Theorem ($\mathcal{L}_L$-interpolation conditions)**

*$\left\{(x_1, y_1), \dots, (x_N, y_N)\right\}$ and $\left\{(u_1, v_1), \dots, (u_N, v_N)\right\}$ are **$\mathcal{L}_L$-interpolable** if, and only if,*

$$\begin{cases} X^T V = Y^T U, \\ Y^T Y \preceq L^2 X^T X, \\ V^T V \preceq L^2 U^T U, \end{cases}$$

*where $X = (x_1 \ \cdots \ x_N)$, $Y = (y_1 \ \cdots \ y_N)$, $U = (u_1 \ \cdots \ u_N)$ and $V = (v_1 \ \cdots \ v_N)$.*

**Definition ($\mathcal{S}_{0,L}$-interpolability)**

$\left\{(x_1, y_1), \ldots, (x_N, y_N)\right\}$ is **$\mathcal{S}_{0,L}$-interpolable** if, and only if,

$$\exists Q \text{ symmetric with } 0 \preceq Q \preceq LI \;:\; y_i = Qx_i \quad \forall i.$$

**Theorem ($\mathcal{S}_{0,L}$-interpolation conditions)**

*$\left\{(x_1, y_1), \ldots, (x_N, y_N)\right\}$ is **$\mathcal{S}_{0,L}$-interpolable** if, and only if,*

$$\begin{cases} X^T Y = Y^T X, \\ Y^T(LX - Y) \succeq 0, \end{cases}$$

*where $X = (x_1 \;\cdots\; x_N)$ and $Y = (y_1 \;\cdots\; y_N)$.*

**Remark:** Similar result for skew-symmetric matrices.

**Definition ($\mathcal{S}_{\mu,L}$-interpolability)**

$\left\{(x_1, y_1), \ldots, (x_N, y_N)\right\}$ is **$\mathcal{S}_{\mu,L}$-interpolable** if, and only if,

$$\exists Q \text{ symmetric with } \mu I \preceq Q \preceq LI \;:\; y_i = Qx_i \quad \forall i.$$

**Theorem ($\mathcal{S}_{\mu,L}$-interpolation conditions)**

*$\left\{(x_1, y_1), \ldots, (x_N, y_N)\right\}$ is **$\mathcal{S}_{\mu,L}$-interpolable** if, and only if,*

$$\begin{cases} X^T Y = Y^T X, \\ (Y - \mu X)^T (LX - Y) \succeq 0, \end{cases}$$

*where $X = (x_1 \;\cdots\; x_N)$ and $Y = (y_1 \;\cdots\; y_N)$.*

**Remark:** Similar result for skew-symmetric matrices.

## Interpolation conditions for homogeneous quadratic functions $\mathcal{Q}_{\mu,L} = \left\{f(x) = \frac{1}{2}x^TQx, Q = Q^T, \mu I \preceq Q \preceq LI\right\}$

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$$\exists Q \text{ symmetric with } \mu I \preceq Q \preceq LI \ : \ \begin{cases} f_i = \frac{1}{2}x_i^TQx_i, & \forall i, \\ g_i = Qx_i, & \forall i. \end{cases}$$

**Theorem ($\mathcal{Q}_{\mu,L}$-interpolation conditions)**

*$\left\{(x_1, g_1, f_1), \ldots, (x_N, g_N, f_N)\right\}$ is **$\mathcal{Q}_{\mu,L}$-interpolable** if, and only if,*

$$\begin{cases} X^TG = G^TX, \\ (G - \mu X)^T(LX - G) \succeq 0, \\ f_i = \frac{1}{2}x_i^Tg_i \quad \forall i, \end{cases}$$

*where $X = (x_1 \ \cdots \ x_N)$ and $G = (g_1 \ \cdots \ g_N)$.*

# Interpolation conditions for homogeneous quadratic functions

$\mathcal{Q}_{\mu,L} = \left\{f(x) = \frac{1}{2}x^TQx, Q = Q^T, \mu I \preceq Q \preceq LI\right\}$

**Definition ($\mathcal{Q}_{\mu,L}$-interpolability)**

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Allows to use PEP on the class of quadratic functions!

# Exploitation of linear operators

## interpolations

1 Gradient method on $g(Mx)$

2 Gradient method on $\frac{1}{2}x^T Qx$

3 Chambolle-Pock method on $f(x) + g(Mx)$

# Simplest problem $\min_x F(x)$ involving linear operator

**Function class:** $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx), g \text{ is } L_g\text{-smooth } \mu_g\text{-strongly convex}, 0 \leq ||M|| \leq L_M\}$
**Gradient method (GM):** $x_{i+1} = x_i - \frac{h}{L_g L_M^2} \nabla F(x_i)$

**Function class:** $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx),\ g \text{ is } L_g\text{-smooth } \mu_g\text{-strongly convex},\ 0 \leq ||M|| \leq L_M\}$

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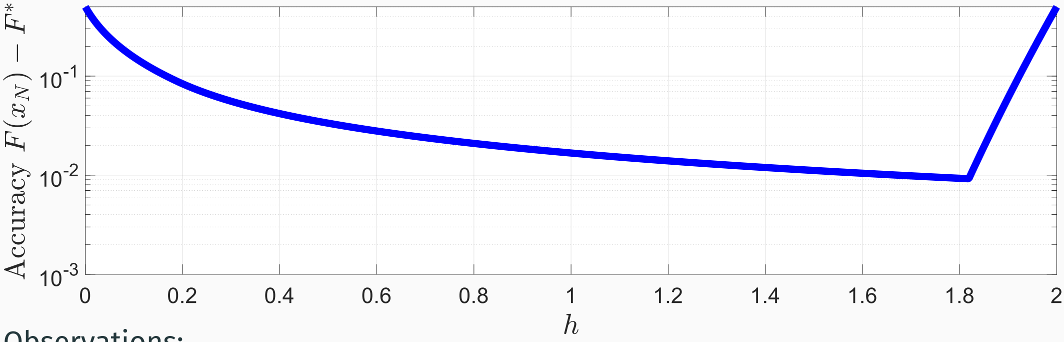
**Parameters:** $L_g = L_M = 1$, $\mu_g = 0.1$, $\kappa_g = \frac{\mu_g}{L_g}$, $||x_0 - x^*||^2 \leq 1$, $h_0(\kappa_g, N)$

**Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$**




**Observations:**

# Simplest problem $\min_x F(x)$ involving linear operator

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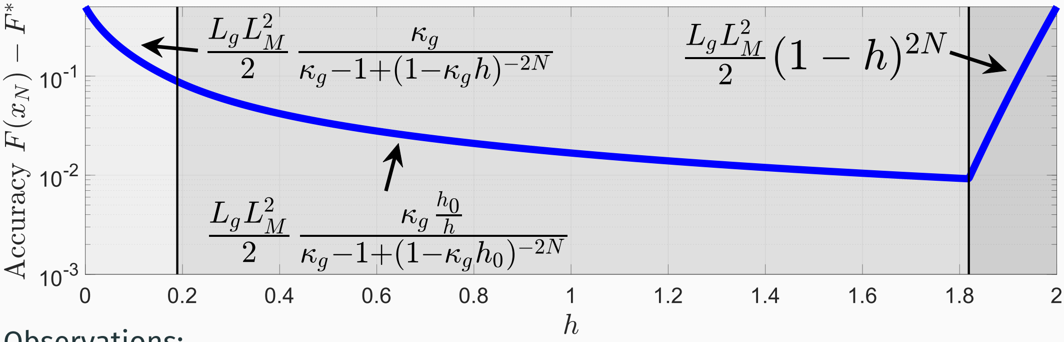


**Observations:**

- 3 (identified) regimes;

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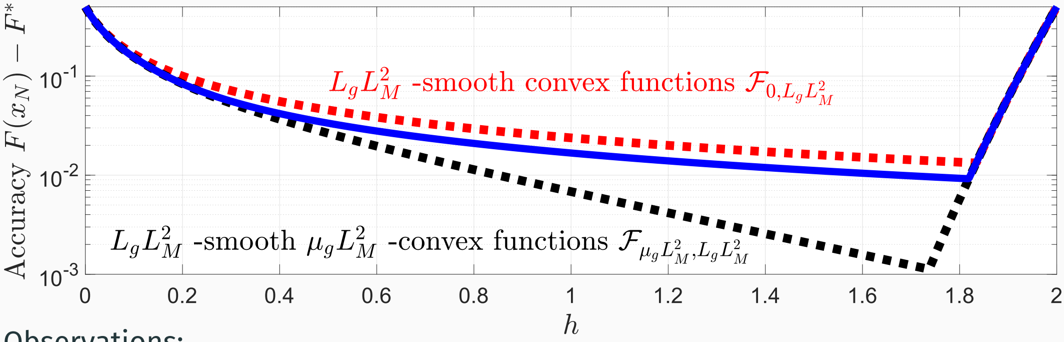


**Observations:**

- 3 (identified) regimes;
- $\mathrm{Acc}(\mathcal{F}_{\mu_g L_M^2, L_g L_M^2}) \leq \mathrm{Acc}(\mathcal{C}^{0,L_M}_{\mu_g,L_g}) \leq \mathrm{Acc}(\mathcal{F}_{0,L_g L_M^2})$;

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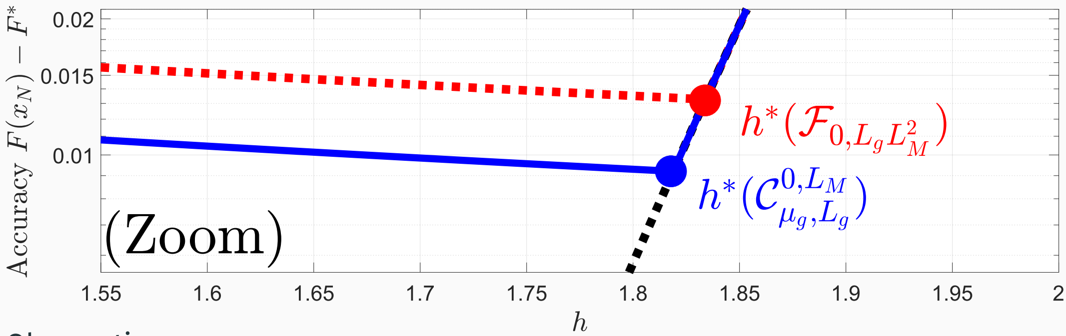


**Observations:**

- 3 (identified) regimes;
- $\mathrm{Acc}(\mathcal{F}_{\mu_g L_M^2, L_g L_M^2}) \leq \mathrm{Acc}(\mathcal{C}^{0,L_M}_{\mu_g,L_g}) \leq \mathrm{Acc}(\mathcal{F}_{0,L_g L_M^2})$;
- Information on structure improves optimal step;

Function class: $\mathcal{C}^{0,L_M}_{\mu_g,L_g} = \{F(x) = g(Mx),\ g$ is $L_g$-smooth $\mu_g$-strongly convex, $0 \leq ||M|| \leq L_M\}$

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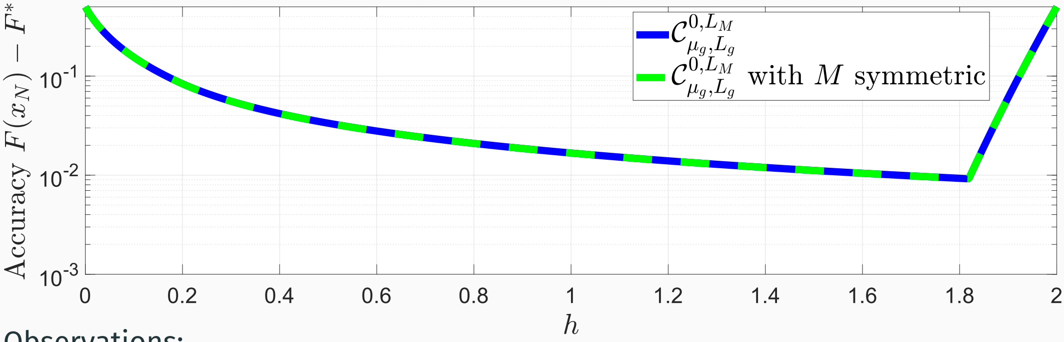
Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$




Observations:

- 3 (identified) regimes;
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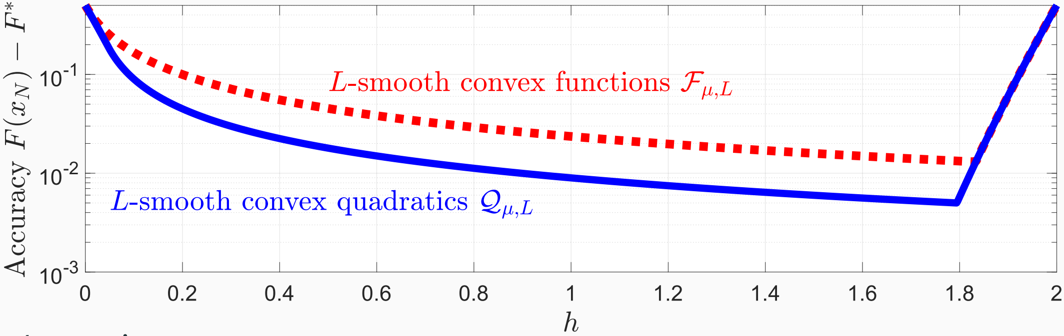
- Information on structure improves optimal step;
- Symmetry has no impact.

Function class: $\mathcal{Q}_{\mu,L} = \{F(x) = \frac{1}{2}x^TQx,\ \mu I \preceq Q \preceq LI,\ Q = Q^T\}(= \mathcal{D}_{1,1}^{\sqrt{\mu},\sqrt{L}})$

Gradient method (GM): $x_{i+1} = x_i - \frac{h}{L}\nabla F(x_i)$

Notations/parameters: $L = 1$, $\mu = 0$, $||x_0 - x^*||^2 \leq 1$

Worst-case performance of $N = 10$ iterations (GM) for varying $h \in [0, 2]$




Observations:

- 3 (identified) regimes;
- $\mathrm{Acc}(\mathcal{Q}_{\mu,L}) \leq \mathrm{Acc}(\mathcal{F}_{\mu,L})$;
- Information on structure improves optimal step.

**Problem:** $\min_x f(x) + g(Mx)$ where $f, g$ convex, proximable, $0 \leq ||M|| \leq L_M$

**Chambolle-Pock (CP) method:**
$$\begin{cases} x_{i+1} &= \mathrm{prox}_{\tau f(.)}\left(x_i - \tau M^T u_i\right) \\ u_{i+1} &= \mathrm{prox}_{\sigma g^*(.)}\left(u_i + \sigma M(2x_{i+1} - x_i)\right) \end{cases}$$

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Convergence results available but implicit and technical assumptions

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## Examples of assumptions:

- Existence of sets containing all iterations [Chambolle and Pock, 2011];
- Bound depending on the instance of the problem [Chambolle and Pock, 2016];
- Bound depending on the distance from initial to last point [Amir Beck, 2022].

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> PEP uses classical explicit assumptions to unify performance guarantees.

**Problem:** $\min_x f(x) + g(Mx)$ where $f, g$ convex, proximable, $0 \leq ||M|| \leq L_M$, bounded subgradient

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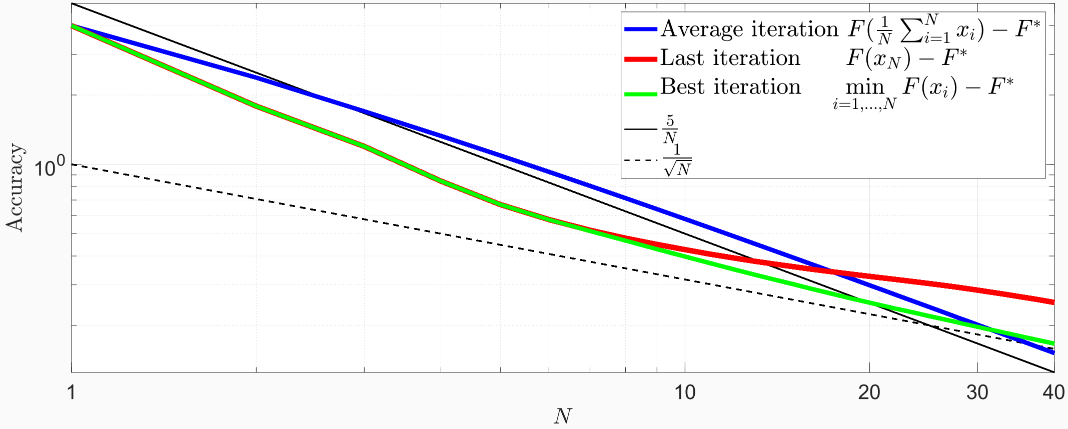
**Parameters:** $L_M = \tau = \sigma = 1$, $\mu_M = 0$, $||x_0 - x^*|| \leq 1$, $||u_0 - u^*||^2 \leq 1$, $F(x) = f(x) + g(Mx)$

Worst-case performance of (CP) for varying number of iterations $N$



**State-of-the-art**

- PEP gives exact worst-case **performance of optimization methods**

**Contribution**

- Extend PEP to methods/functions involving **linear operators** (by obtaining **interpolation conditions**)
- Analyze **Gradient and Chambolle-Pock methods**

**Future steps**

- Analyze methods (e.g. PDFP, CV, PD30, ADMM) on

$$F_1(x) = f(x) + g(Mx), \qquad F_2(x) = f(x) + \frac{1}{2}x^T Q x, \qquad F_3(x) = g(Mx)$$

- Represent Hessian as linear operator to analyze **second-order methods**

Interpolation Conditions for Linear Operators and Applications to PEP

Preprint: *https://arxiv.org/abs/2302.08781*

Nizar Bousselmi (*nizar.bousselmi@uclouvain.be*)

# Worst-case performance of (GM) on $\mathcal{F}_{\mu,L}$ and $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$

## Worst-case performance of (GM) on $\mathcal{F}_{\mu,L}$

$$F(x_N) - F^* \leq \frac{LR^2}{2} \max\left\{ \frac{\kappa}{\kappa - 1 + (1-\kappa h)^{-2N}}, (1-h)^{2N} \right\}$$

where $\kappa = \frac{\mu}{L}$.

## Worst-case performance of (GM) on $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$

$$F(x_N) - F^* \leq \frac{L_g L_M^2 R^2}{2} \max\left\{ \frac{\kappa_g \alpha}{\kappa_g - 1 + (1-\kappa_g \alpha h)^{-2N}}, (1-h)^{2N} \right\}$$

where $\kappa_g = \frac{\mu_g}{L_g}$, $\kappa_M = \frac{\mu_M}{L_M}$ and $\alpha = \mathrm{proj}_{[\kappa_M^2,1]}\left(\frac{h_0}{h}\right)$ for $h_0$ solution of

$$\begin{cases} (1-\mu_g)(1-\mu_g h_0)^{2N+1} = 1 - (2N+1)\mu_g h_0 \\ 0 \leq h_0 \leq \frac{1}{\mu_g}. \end{cases}$$

# Worst-case functions of (GM) on $\mathcal{F}_{\mu,L}$ and $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$

Let $q(x) = \frac{1}{2}x^2$ and $\ell_{\mu,h}(x) = \begin{cases} \frac{\mu}{2}x^2 + (1-\mu)\tau_{\mu,h}|x| - \left(\frac{1-\mu}{2}\right)\tau_{\mu,h}^2 & \text{if } |x| \geq \tau_{\mu,h}, \\ \frac{1}{2}x^2 & \text{else,} \end{cases}$ where $\tau_{\mu,h} = \frac{\mu}{\mu-1+(1-\mu h)^{-2N}}$.

## Worst-case functions of (GM) on $\mathcal{F}_{\mu,L}$ $(\kappa = \frac{\mu}{L})$ [Taylor, Hendrickx, Glineur, 2017]

$Lq(x) = \frac{L}{2}x^2$ and $L\ell_{\kappa,h}(x) = \begin{cases} \frac{\mu}{2}x^2 + (L-\mu)\tau_{\kappa,h}|x| - \left(\frac{L-\mu}{2}\right)\tau_{\kappa,h}^2 & \text{if } |x| \geq \tau_{\kappa,h}, \\ \frac{L}{2}x^2 & \text{else,} \end{cases}$

where $\tau_{\kappa,h} = \frac{\kappa}{\kappa-1+(1-\kappa h)^{-2N}}$.

## Worst-case functions of (GM) on $\mathcal{S}_{\mu_g,L_g}^{\mu_M,L_M}$ $(\kappa_g = \frac{\mu_g}{L_g}, \kappa_M = \frac{\mu_M}{L_M})$ [Bousselmi, Hendrickx, Glineur, 2023]

$L_g L_M q(x) = \frac{L_g}{2}(L_M x)^2$ and

$$L_g L_M M^2 \ell_{\kappa_g,M^2h}(x) = \begin{cases} \frac{\mu_g}{2}(L_M M x)^2 + (L_g - \mu_g)L_M M \tau_{\kappa_g,M^2h}|L_M M x| - \left(\frac{L_g-\mu_g}{2}\right) L_M M \tau_{\kappa_g,M^2h}^2 & \text{if } |x| \geq \tau_{\kappa_g,M^2h}, \\ \frac{L_g}{2}(L_M M x)^2 & \text{else,} \end{cases}$$

where $\tau_{\kappa_g,M^2h} = \frac{\kappa_g}{\kappa_g-1+\left(1-\kappa_g M^2 h\right)^{-2N}}$ and $M = \mathrm{Proj}_{[\kappa_M,1]}\left(\sqrt{\frac{h_0}{h}}\right)$.

- $\ell_p$-regularized robust regression ($p = 1, 2$)

$$\min_x ||Mx - b||_1 + ||x||_p^p,$$

- $\ell_1$-constrained least squares

$$\min_x \ ||Mx - b||_2^2,$$
$$||x||_1 \leq c,$$

- Basis pursuit

$$\min_x \ ||x||_1,$$
$$Mx = c,$$

- Total variation deblurring

$$\min_x \ ||M_1 x - b||_2^2 + ||M_2 x||_1,$$

- Resource allocation

$$\min_x \ F(x),$$
$$Mx = c.$$

# Methods with linear operators ($\min_x f(x) + g(Mx) + h(x)$, $g, h$ proximable and $f$ smooth)

- **Primal-Dual Fixed Point (PDFP)** [Chen et al., 2016]

$$\begin{cases} \tilde{x} = \mathrm{prox}_{\tau h}\left(x_i - \tau \nabla f(x_i) - \tau M^T u_i\right), \\ u = \mathrm{prox}_{\sigma g^*}\left(u_i + \sigma M \tilde{x}\right), \\ x = \mathrm{prox}_{\tau h}\left(x_i - \tau \nabla f(x_i) - \tau M^T u\right), \\ x_{i+1} = x_i + \rho_i (x - x_i), \\ u_{i+1} = u_i + \rho_i (u - u_i). \end{cases}$$

- **Condat-Vu (CV)** [Condat, 2013, Vu, 2013]

$$\begin{cases} x = \mathrm{prox}_{\tau h}\left(x_i - \tau \nabla f(x_i) - \tau M^T u_i\right), \\ u = \mathrm{prox}_{\sigma g^*}\left(u_i + \sigma M(2x - x_i)\right), \\ x_{i+1} = x_i + \rho_i (x - x_i), \\ u_{i+1} = u_i + \rho_i (u - u_i). \end{cases}$$

- **Primal-Dual Three-Operator Splitting (PD3O)**

$$\begin{cases} x = \mathrm{prox}_{\tau h}\left(x_i\right), \\ u = \mathrm{prox}_{\sigma g^*}\left(u_i + \sigma M(2x - x_i - \tau \nabla f(x) - \tau M^T u_i)\right), \\ x_{i+1} = x_i + \rho_i (x - x_i - \tau \nabla f(x) - \tau M^T u), \\ u_{i+1} = u_i + \rho_i (u - u_i). \end{cases}$$

- **Alternating Direction Method of Multipliers (ADMM)** [Gabay and Mercier, 1976]:

$$\begin{cases} x_{i+1} \in \arg\min_x f(x) + \frac{\rho}{2} M_1 x + M_2 y_i - c + \frac{1}{\rho} z_i^2, \\ y_{i+1} \in \arg\min_y g(y) + \frac{\rho}{2} M_1 x_{i+1} + M_2 y - c + \frac{1}{\rho} z_i^2, \\ z_{i+1} = z_i + \rho (M_1 x_{i+1} + M_2 y_{i+1} - c), \end{cases}$$

solves $\min_x f(x) + g(y)$ s.t. $M_1 x + M_2 y = c$.

- **Proximal Alternating Predictor-Corrector (PAPC)** [Drori et al., 2015]

$$\begin{cases} p_{i+1} = x_i - \tau\left(M y_i + \nabla f(x_i)\right), \\ y_{i+1} = \mathrm{prox}_h\left(y_i + M^T p_{i+1}\right), \\ x_{i+1} = x_i - \tau\left(M y_{i+1} + \nabla f(x_i)\right), \end{cases}$$

solves $\min_x \max_y f(x) + x^T M y - h(y)$.

# Proof of necessity

**Theorem ($L$-matrix-interpolation conditions)**

*$\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ are **L-matrix-interpolable** if, and only if,*

$$\begin{cases} X^TV = Y^TU, \\ Y^TY \preceq L^2X^TX, \\ V^TV \preceq L^2U^TU, \end{cases}$$

*where $X = (x_1 \ \cdots \ x_{N_1})$, $Y = (y_1 \ \cdots \ y_{N_1})$, $U = (u_1 \ \cdots \ u_{N_2})$ and $V = (v_1 \ \cdots \ v_{N_2})$.*

Proof.
**Necessity**: Let $\{(x_i, y_i)\}_{i=1,\dots,N_1}$ and $\{(u_j, v_j)\}_{j=1,\dots,N_2}$ $L$-matrix-interpolable.

$$\Leftrightarrow \exists M \text{ with } \sigma_{\max}(M) \leq L \ : \qquad y_i = Mx_i \text{ and } v_j = M^Tu_j \ \ \forall i, j$$

$$\Leftrightarrow \exists M \text{ with } M^TM \preceq L^2I \text{ and } MM^T \preceq L^2I \ : \ Y = MX \text{ and } V = M^TU$$

$$\Rightarrow \underbrace{X^TM^TU}_{X^TV} = \underbrace{X^TM^TU}_{Y^TU}, \qquad \underbrace{X^TM^TMX}_{Y^TY} \preceq L^2X^TX, \qquad \underbrace{U^TMM^TU}_{V^TV} \preceq L^2U^TU$$

□

## Proof of sufficiency in short

Let $(X, Y, U, V)$ satisfying $\begin{cases} X^T V &= Y^T U, \\ Y^T Y &\preceq L^2 X^T X, \\ V^T V &\preceq L^2 U^T U, \end{cases}$ then,

- **Step 1:** $\exists (X_R, Y_R, U_R, V_R)$ building the same Gram matrices, i.e.

$$(X\ V)^T (X\ V) = (X_R\ V_R)^T (X_R\ V_R),$$
$$(Y\ U)^T (Y\ U) = (Y_R\ U_R)^T (Y_R\ U_R),$$

and such that $\exists M_R$ with $\sigma_{\max}(M_R) \leq L$ : $\begin{cases} Y_R = M_R X_R, \\ V_R = M_R^T U_R. \end{cases}$

- **Step 2:** If $(X, Y, U, V)$ and $(X_R, Y_R, U_R, V_R)$ build the same Gram matrices, then,

$\exists R_1, R_2$ unitary : $\begin{cases} (X_R\ V_R) = R_1 (X\ V), \\ (Y_R\ U_R) = R_2 (Y\ U), \end{cases}$ therefore, $\begin{cases} Y = \overbrace{R_2^T M_R R_1}^{M} X, \\ V = \overbrace{R_1^T M_R^T R_2}^{M^T} U. \end{cases}$