PERFORMANCE ESTIMATION OF FIRST-ORDER OPTIMIZATION METHODS ON CONVEX FUNCTIONS COMPOSED WITH LINEAR MAPPINGS

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Worst-case performance of a method on a class of functions

Common question in optimization :

Worst-case **performance** of an optimization method $\mathcal{M}$ on

$$\min_x f(x)$$

where $f \in \mathcal{F}$ has some properties (smoothness, convexity,...) ?

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Example :

Worst-case performance of $\overbrace{\text{gradient method}}^{\mathcal{M}}$ on $\overbrace{L\text{-smooth convex functions}}^{\mathcal{F}}$ after N iterations ?

$$\overbrace{f(x_N) - f^*}^{\text{performance}} \leq \frac{L}{2} \frac{1}{2N+1}.$$

Performance Estimation Problem (PEP)

Theoretical and practical framework to analyze **performance** of optimization methods on problem classes.

- *Performance of first-order methods...*Drori & Teboulle 2013
- *Convex interpolation and performance estimation...*Taylor 2017

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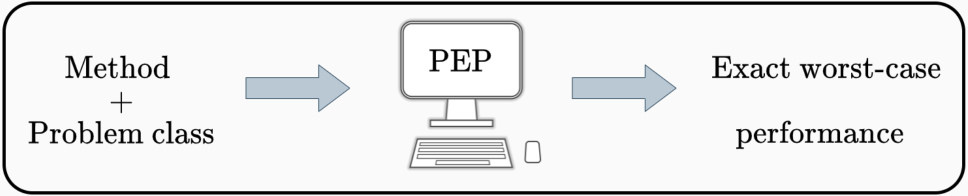

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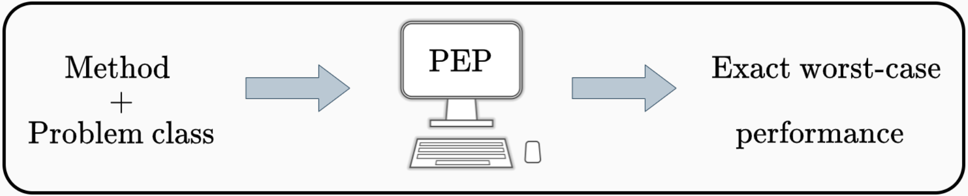

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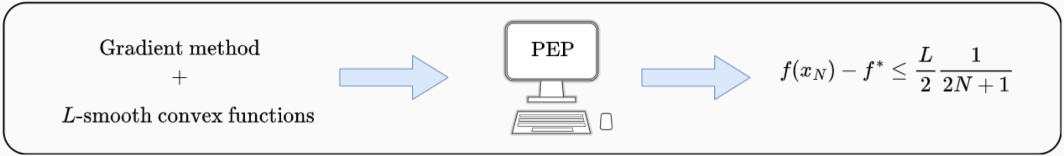

PEP IS WELL DEVELOPED

Large number of methods and function classes already analyzed through PEP

- **Smooth convex** and **smooth strongly convex** functions; [Taylor, Hendrickx, Glineur]
- **Constrained** optimization (projected gradient); [Taylor, Glineur, Hendrickx]
- **Non-smooth** optimization (subgradient, proximal operators); [Taylor, Glineur, Hendrickx]
- **Non-convex** and **hypo-convex** functions [Rotaru, Glineur, Patrinos], [Abbaszadehpeivasti, de Klerk, Zamani]
- **Stochastic** optimization;
- **Decentralized** optimization; [Colla, Hendrickx]
- **Coordinate descent** method;
- etc.

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- **Coordinate descent** method;
- etc.
- Our contribution: Convex functions composed with **linear mappings**.

Performance Estimation Problem

Interpolation conditions for linear mappings

Exploitation of new tool

Input = optimization method + problem class
Output = worst instance in problem class

Example:

- N steps of gradient method $x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k)$
- L-smooth convex functions f

PEP

$$
\begin{aligned}
\max_{\text{points } x_k, x^*, \text{ function } f} \quad & f(x_N) - f(x^*) \\
\text{s.t.} \quad & f \text{ } L\text{-smooth convex}, \\
& x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k), \\
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& \nabla f(x^*) = 0.
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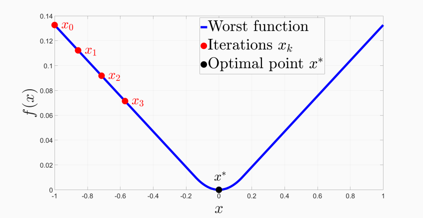
PEP solved

Output :

- Worst performance : $f(x_N) - f^* \leq \frac{L}{2}\frac{1}{2N+1}$ for any N;
- Worst function for $N = 3$ and $L = 1$:



PEP as finite-dimensional problem

f **infinite-dimensional** but only access to x_k, $f(x_k)$, $\nabla f(x_k)$ … **black-box** property !

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$$\max_{\text{points } x_k, x^*, f_k, f^*, g_k, g^*} \quad f_N - f^*$$

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\text{s.t.} \quad & \exists f \ L\text{-smooth convex}: \quad f(x_k) = f_k, \quad \nabla f(x_k) = g_k, \\
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Interpolation condition to reformulate.

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$$\max_{\text{points } x_k, x^*, f_k, f^*, g_k, g^*} \quad f_N - f^*$$

$$\text{s.t.} \quad f_j \geq f_k + g_k^T(x_j - x_k) + \frac{1}{2L}||g_j - g_k||^2,$$

$$x_{k+1} = x_k - \frac{1}{L} g_k,$$

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Interpolation condition to reformulate.

Can be reformulated as convex **semidefinite problem**, efficiently solvable !

Interpolation conditions for L-smooth convex functions

Given x_k, g_k and f_k $\quad \forall k = 0, \dots, N,$

$\exists$ L-smooth convex f such that $\begin{cases} f(x_k) & = f_k \;\; \forall k = 0, \dots, N, \\ \nabla f(x_k) & = g_k \;\; \forall k = 0, \dots, N, \end{cases}$ if and only if

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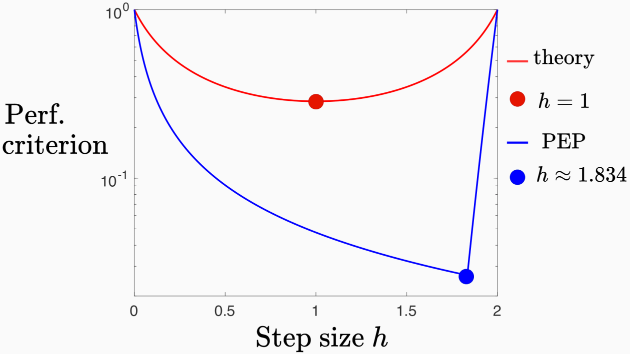
Remark : Interpolation conditions (and PEP formulation) exist for numerous function classes : **non-smooth, L-smooth, convex, μ-strongly convex,etc**

Accuracy after 10 steps of gradient method on L-smooth convex functions for varying step size $\frac{h}{L}$



Remarks :

- Theory [Nesterov98] suggests a step size of $\frac{1}{L}$ while PEP recommends $\approx \frac{1.834}{L}$ to optimize the worst-case;
- PEP provides tight results;
- PEP guarantees a performance $\approx$ 10 times better than theory.

Outline

Performance Estimation Problem

Interpolation conditions for linear mappings

Exploitation of new tool

Goal : Analyze worst performance of methods on $\min_x f(x)$ for f using linear mapping.

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Function class	Hypothesis
$f(x) = g(Ax)$	g smooth (strongly) convex
$f(x) = g(Ax) = \frac{1}{2}x^T QX$ (quadratic functions)	$g(y) = \frac{1}{2}\|\|y\|\|^2$ and $Q = A^T A$
$f(x) = h(x) + g(Ax)$	h, g smooth (strongly) convex

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or equivalently, decomposing operations for each step

$$
\begin{cases}
y_k & = Ax_k \\
u_k & = \nabla g(y_k) \\
v_k & = Au_k \\
x_{k+1} & = x_k - \frac{h}{L} v_k
\end{cases}
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$$\begin{cases} y_k &= Ax_k \quad \text{New interpolation conditions} \\ u_k &= \nabla g(y_k) \quad \text{Standard interpolation conditions} \\ v_k &= Au_k \quad \text{New interpolation conditions} \\ x_{k+1} &= x_k - \frac{h}{L} v_k \quad \text{Standard} \end{cases}$$

Let $N \in \mathbb{N}$, $S = \{0, \dots, N\}$, $X = (x_0 \ \cdots \ x_N)$, $Y = (y_0 \ \cdots \ y_N)$ and $L \in \mathbb{R}$.

Theorem (Symmetric matrix with spectrum between 0 and L)

Given x_k and y_k $\ \forall k \in S$ and $G = (X\ Y)^T (X\ Y) = \begin{pmatrix} X^T X & X^T Y \\ Y^T X & Y^T Y \end{pmatrix} \triangleq \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0.$

If $\exists Q$ symmetric $\ : \ 0 \preceq Q \preceq L$ and $y_k = Q x_k \ \forall k = 0, \dots, N$ then

$$\begin{cases} B = B^T, \\ B \succeq \frac{C}{L}. \end{cases}$$

Let $N \in \mathbb{N}$, $S = \{0, \ldots, N\}$, $X = (x_0 \ \cdots \ x_N)$, $Y = (y_0 \ \cdots \ y_N)$ and $\mu \leq L \in \mathbb{R}$.

Theorem (Symmetric matrix with spectrum between μ and L)

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If $\exists Q$ symmetric $: \mu I \preceq Q \preceq LI$ *and* $y_k = Qx_k \ \ \forall k = 0, \ldots, N$ *then*

$$\begin{cases} B = B^T, \\ B \succeq \frac{\mu L}{\mu+L} A + \frac{1}{\mu+L} C. \end{cases}$$

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Remark :

- We can have the «**only if**» with a slightly modified theorem;
- We only consider homogeneous quadratic functions;
- A similar theorem exists for **non-symmetric** matrix with bounded singular values.

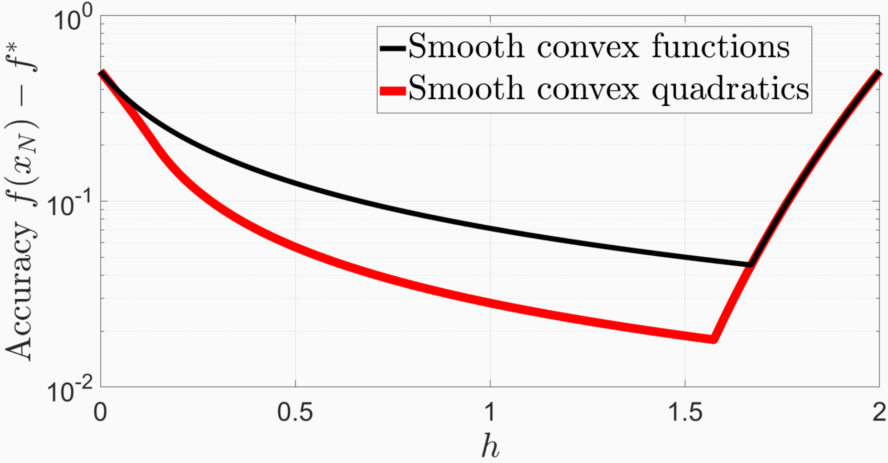
Performance Estimation Problem

Interpolation conditions for linear mappings

Exploitation of new tool

Worst performance of $N = 3$ steps of gradient method on $\min_x f(x)$ w.r.t. the step size h

- $f(x) = g(Ax) = \frac{1}{2}x^TQx$, $g(y) = \frac{1}{2}||y||^2$
- $Q = A^TA$ with $0 \preceq Q \preceq I$



Smooth convex functions	Smooth convex quadratic
$f(x_N) - f^* \leq \frac{LR^2}{2} \begin{cases} \frac{1}{2Nh+1} & \text{if } h \in [0, h_0] \\ (1-h)^{2N} & \text{if } h \in [h_0, \infty] \end{cases}$	$f(x_N) - f^* \leq \frac{LR^2}{2} \begin{cases} (1-h)^{2N} & \text{if } h \in [0, \frac{1}{2N+1}] \\ \frac{1}{h}\frac{(2N)^{2N}}{(2N+1)^{2N+1}} & \text{if } h \in [\frac{1}{2N+1}, h_1] \\ (1-h)^{2N} & \text{if } h \in [h_1, \infty] \end{cases}$

Note : As in the general case, worst quadratic functions are one-dimensional.

Independent existing approach to analyze first-order methods on quadratic functions

Worst-case performance of a first-order method on quadratic functions is

Polynomial approach

$$\max_{\rho\in[\mu,L]} \frac{\rho}{2} R\big(1+\rho K_N(\rho)\big)^2$$

where K_N is a polynomial of degree N that depends explicitly on the method.

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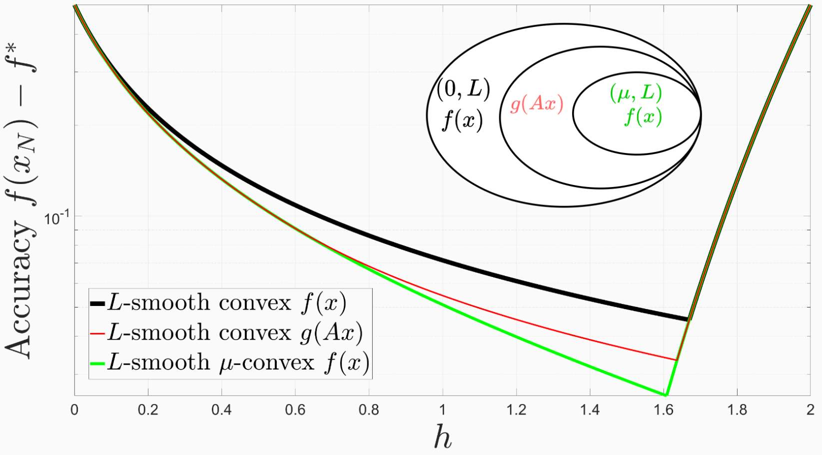
where K_N is a polynomial of degree N that depends explicitly on the method.

It only works for quadratic functions !

Function class	PEP	Polynomial approach
Quadratics $\frac{1}{2}x^T Q x$	OK	OK
g(Ax)	OK	KO
f(x) + g(Ax)	OK	KO
Class with matrices (except quadratics)	OK	KO

Worst performance of $N = 3$ steps of gradient method on $\min_x f(x)$ w.r.t. the step size h

- $f(x) = \mathbf{g(Ax)}$, $g(y)$ smooth strongly convex
- $0 \preceq A \preceq I$



Smooth convex $f(x)$	Smooth convex $g(Ax)$
$\frac{LR^2}{2}\begin{cases}\frac{1}{2Nh+1} & \text{if } h\in[0,h_0]\\ (1-h)^{2N} & \text{if } h\in[h_0,\infty]\end{cases}$	$\frac{LR^2}{2}\begin{cases}\frac{\kappa_g}{\kappa_g-1+(1-\kappa_g h)^{-2N}} & \text{if } h\in[0,h_2]\\ \frac{\kappa_g\frac{h_0}{h}}{\kappa_g-1+(1-\kappa_g h_0)^{-2N}} & \text{if } h\in[h_2,h_3]\\ (1-h)^{2N} & \text{if } h\in[h_3,\infty]\end{cases}$

State of the art :

- *PEP* gives **worst-case performance** of **methods** on a **function class** (for which interpolation conditions are available).

Our contribution :

- Extending PEP to methods and classes using linear mapping;
- Analyzing $\frac{1}{2}x^T QX$, $g(Ax)$.

Future research : Analyzing more complex problems and methods (e.g. $f(x) + g(Ax)$, Chambolle-Pock, Condat-Vu) and identifying performance.

Definitions and notations

f is L-smooth when

$$||\nabla f(x) - \nabla f(y)|| \leq L||x - y||.$$

First-order method of the form

$$x_N = x_0 - \sum_{i=0}^{N-1} h_{N,i} \nabla f(x_i).$$

Case $\mu = L$

Let $G = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \succeq 0$ and $\mu = L \in \mathbb{R}$.

Theorem

G can be written as $\begin{pmatrix} X^TX & X^TQX \\ X^TQX & X^TQ^2X \end{pmatrix}$ for a symmetric matrix Q with $LI \preceq Q \preceq LI$ if and only if

$$B = B^T,$$
$$C \preceq L^2 A.$$

Interpolation condition for L-smooth convex functions

f L-smooth convex if and only if

$$f(x) \geq f(y) + \nabla f^T(y)(x-y) + \frac{1}{2L}||\nabla f(x) - \nabla f(y)||^2 \quad \forall x, y$$

f L-smooth convex : $f(x_k) = f_k, \; \nabla f(x_k) = g_k$ if and only if

$$f_i \geq f_j + g_j^T(x_i - x_j) + \frac{1}{2L}||g_i - g_j||^2 \quad \forall i, j$$

SDP formulation

N steps of gradient method on L-smooth convex functions.

Matrix variable: $G = (g_0 \ \dots \ g_N \ x_0)^T (g_0 \ \dots \ g_N \ x_0) \in \mathbb{S}^{N+2}$

Parameters:

- $h_i = (0 \ \dots 0 \ \frac{-1}{L} \ 0 \ \dots \ 0 \ 1) \in \mathbb{R}^{N+2}$
- $u_i = (0 \dots 0 \ 1 \ 0 \ \dots 0) \in \mathbb{R}^{N+2}$
- $2A_{ij} = u_j(h_i - h_j)^T + (h_i - h_j)u_j^T + \frac{1}{L}(u_i - u_j)(u_i - u_j)^T$
- $A_R = u_{N+1}u_{N+1}^T$

$$
\begin{aligned}
\max_{G \in \mathbb{S}^{N+2}, f \in \mathbb{R}^{N+1}} \quad & f_N - f^* \\
\text{s.t.} \quad & f_j - f_i + \mathrm{Tr}(GA_{ij}) \leq 0, \quad \forall i, j \\
& \mathrm{Tr}(GA_{ij}) - R^2 \leq 0, \quad \forall i, j \\
& G \succeq 0.
\end{aligned}
$$